Physics I
ISI B.Math
Mid Semestral Exam : September 28, 2011
Full Marks :80

Answer any five questions. Each question carries 16 marks.

1. A particle P of unit mass moves on the positive x-axis under the force field

$$F = \frac{36}{x^3} - \frac{9}{x^2}$$

where $x > 0$.
a) Show that each motion of P consists of either (i) a periodic oscillation between two extreme points, or(ii) an unbounded motion with one extreme point, depending upon the value of the total energy. (6)
b) Initially P is projected from the point $x = 4$ with speed 0.5. Show that P oscillates between two extreme points and find the period of the motion. [You may make use of the formula $\int_a^b \frac{xdx}{[(x-a)(b-x)]^{\frac{1}{2}}} = \frac{\pi(a+b)}{2}$.](6)
c)Show that there is a single equilibrium position for P and that it is stable. Find the period of small oscillations about this point.(4)

2. (a) A hill is sloped downward at an angle θ with the horizontal. A projectile of mass m is fired with speed v_0 perpendicular to the hill. When it eventually lands on the hill let its velocity make an angle β with the horizontal. Which of the quantities θ, m, v_0 and g does the angle depend on ? Give reasons for your answer. (3)
(b) A beach ball is thrown upward with an initial speed v_0. Assume that the drag force from the air is $F_d = -m\alpha v$.
(i) Show that the maximum height achieved by the ball is given by

$$h = \frac{v_0}{\alpha} - \frac{g}{\alpha^2} \ln\left(1 + \frac{\alpha v_0}{g}\right)$$

Also show that this reduces to the usual expression for maximum height in vacuum in the limit of negligible drag.(4)
(ii)What is the speed v_f of the beach ball right before it hits the ground ? (An implicit equation for v_f is sufficient).(5)
(iii) Does the ball spend more or less time in the air than it would if it were thrown in vacuum? (4)

3. A single particle of mass m, momentum $\mathbf{p}$ and angular momentum $\mathbf{L}$ about the center of force is acted on by an inverse square central force described by $\mathbf{F}(\mathbf{r}) = -\frac{k}{r^2}\hat{\mathbf{r}}$
(a) Show that, the Runge-Lenz-Laplace vector, given by

$$\mathbf{A} = \mathbf{p}\times\mathbf{L} - mk\hat{\mathbf{r}} \tag{1}$$

is conserved during this motion.(5)
(b) In class we have shown that under the above inverse square force , when the total energy $E < 0$, the trajectory of the particle is an ellipse, with the centre of force at one focus of the ellipse. Given this situation, show that the Lenz vector points along the semi-major axis of the ellipse away from the centre.[Hint: Find out the direction of the Lenz vector at the point on the orbit closest to the

focus](5)
(c) Show that the momentum vector moves in a circle of radius $\frac{mk}{L}$ centred on $(0, A/L)$, i.e,

$$p_x^2 + (p_y - A/L)^2 = (mk/L)^2$$

where the x-axis is is chosen along the semi major axis and the z-axis is chosen along L. [Hint: Use an appropriate squared form of equation (1)](6)

4.a) A particle with polar coordinates r, θ which are functions of time t is moving in a plane. The velocity and acceleration of the particle can be written in plane polar coordinates as $\mathbf{v} = v_r \hat{\mathbf{r}} + v_\theta \hat{\theta}$ and $\mathbf{a} = a_r \hat{\mathbf{r}} + a_\theta \hat{\theta}$. Find $v_r, v_\theta, a_r, a_\theta$ as functions of $(r, \theta, \dot{r}, \dot{\theta})$.(8)
b) An insect flies on a spiral trajectory such that its polar coordinates at time t are given by $r = be^{\Omega t}, \theta = \Omega t$. where b and Ω are positive constants. Find the velocity and acceleration vectors of the insect at time t and show that the angle between them is always $\frac{\pi}{4}$(8)

5. a) Consider a mass m acted on by a force $F = -kx$ For a given energy E, plotting x vs $\dot{x}$ will generate a closed curve. Sketch the form of the curve on a x vs $\dot{x}$ plot. Under what conditions will this be a circle?(3)
b)An overdamped harmonic oscillator satisfies the equation

$$\ddot{x} + 10\dot{x} + 16x = 0$$

At time $t = 0$ the particle is projected from the point $x = 1$ toward the origin with speed u. Find x in the subsequent motion.(6)
Show that the particle will reach the origin at some later time t if

$$\frac{u-2}{u-8} = e^{6t}$$

How large must u be so that the particle will pass through the origin? (4 + 3)

6. a) Consider two masses m, connected two each other and to two walls by three springs, as shown in the figure. The three springs have the spring constants $k, 2k$ and $4k$ respectively. Find the most general solution for the positions of the masses as a function of time. What are the normal modes ? Draw two representative sketches at two times for each normal mode to illustrate how the masses move back and forth for each such mode. Is the general motion periodic? (10)
c)A particle of mass m with angular momentum L moves in a spiral trajectory given by $r = r_0 e^{a\theta}$. Find the potential $V(r)$ that leads to this trajectory . (6)

Information you may or may not need:
$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$
$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$
The trajectory of a particle of mass m ,angular momentum L and total energy E moving in the field of force $\mathbf{F}(\mathbf{r}) = -\frac{k}{r^2}\hat{\mathbf{r}}$
is given by

$$\frac{1}{r} = \frac{mk}{L^2}(1 + \epsilon \cos\theta)$$

where $\epsilon = \sqrt{1 + \frac{2EL^2}{mk^2}}$
For a particle of mass m and angular momentum L moving in a central force $\mathbf{f} = f(r)\hat{\mathbf{r}}$,and $u = 1/r$ the path equation is given by

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{L^2u^2} f(\frac{1}{u})$$